

ON THE LTI PROPERTIES OF ADAPTIVE FEEDFORWARD SYSTEMS WITH TAP DELAY-LINE REGRESSORS

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Abstract

It is shown that an adaptive system whose regressor is formed by tap delay-line (TDL) filtering of a multitone sinusoidal signal is representable as a parallel connection of an linear time-invariant (LTI) block and a linear time-varying (LTV) block. A norm-bound (induced 2-norm) is computed explicitly on the LTV block and is shown to decrease as N^{-1} where N is the number of taps. Hence it follows that the adaptive system becomes LTI in the limit as the number of taps goes to infinity. More generally, in the case of finite N , the model can be systematically analyzed using modern robust control methods applicable to LTI systems with norm-bounded LTV perturbations. This analysis puts Glover's 1977 results into a modern robust control setting by providing a valid representation of the adaptive system for any "finite" number of taps, and by exposing the precise nature of convergence to an LTI system as the number of taps becomes large.

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1 INTRODUCTION

In 1977, Glover [3] [10] established an important result that an adaptive feedforward controller whose regressor is formed by tap delay-line (TDL) filtering of a multitone sinusoidal signal can be written as the parallel connection of a linear time-invariant (LTI) and linear time-varying (LTV) subsystem. Glover then makes a heuristic argument that for a large number of taps N the LTI subsystem will dominate the LTV subsystem. This provides a useful approximation because the LTI subsystem has the transfer function of a resonance filter which forms a frequency notch when applied in closed-loop. This provides an important alternative interpretation of the adaptive notching effect associated with the adaptive controller. For example, the LTI interpretation provides estimates of the transients, the depth of the frequency notches, the closed-loop pole locations, etc. all of which are not available from using standard Lyapunov and Hyperstability methods alone [6].

Rigorously speaking, however, Glover's LTI analysis is only applicable in the limiting case of an infinite number of taps $N = \infty$ since it ignores the contribution of the LTV subsystem for any finite number of taps, and does not provide any characterization of the convergence properties as the number of taps increases. Hence, for the realistic case of a finite number of taps, the precise interpretation of Glover's results is unclear. The present paper overcomes these difficulties by putting the problem into a modern robust control setting. Specifically, a norm-bound (the induced 2-norm) is established on the LTV subsystem which is shown to converge to zero at a particular rate as a function of the number of taps, the adaptation gain, the number of tones, and the tone spacing. With this representation, for any finite N , the adaptive system can be rigorously analyzed using robust control methods applicable to LTI systems with norm-bounded LTV perturbations. Furthermore, the choice of N can be determined from precise H_∞ conditions rather than the heuristic "rules of thumb" given in Glover's original paper,

The main results of this paper are taken from a recent JPL internal document [1].

2 BACKGROUND

2.1 Adaptive Systems with Harmonic Regressors

The configuration to be studied is shown in Figure 2.1. An estimate \hat{y} of some signal y is to be constructed as a linear combination of the elements of a regressor vector $x(t) \in \mathbb{R}^N$, i.e.,

Estimated Signal

$$\hat{y} = w(t)^T x(t) \quad (2.1)$$

where $w(t) \in \mathbb{R}^N$ is a parameter vector which is tuned in real-time using the adaptation algorithm,

Adaptation Algorithm

$$w = \mu \Gamma(p)[\tilde{x}(t)e(t)] \quad (2.2)$$

Here, the notation $\Gamma(p)[\cdot]$ is used to denote the multivariable LTI transfer function $\Gamma(s) \cdot I$ where $\Gamma(s)$ is any LTI transfer function in the Laplace s operator (the differential operator p will replace the Laplace operator s in all time-domain filtering expressions); the term $e(t) \in \mathbb{R}^1$ is an error signal; $\mu > 0$ is an adaptation gain; and the signal \tilde{x} is obtained by filtering the regressor x through any stable filter $F(p)$, i.e.,

Regressor Filtering

$$\tilde{x} = F(p)[x] \quad (2.3)$$

The notation $F(p)[\cdot]$ denotes the multivariable LTI transfer function $F(s) \cdot I$ with SISO filter $F(s)$, acting on the indicated vector time domain signal.

For the purposes of this paper, it will be assumed that the regressor x can be written as a linear combination of m distinct sinusoidal components $\{\omega_i\}_{i=1}^m$, $0 < \omega_1 < \omega_2 < \dots < \omega_m$. Equivalently, it is assumed that there exists a matrix $\mathcal{X} \in \mathbb{R}^{N \times 2m}$ such that,

Harmonic Regressor

$$x = xc(t) \quad (2.4)$$

$$c(t) = [\sin(\omega_1 t), \cos(\omega_1 t), \dots, \sin(\omega_m t), \cos(\omega_m t)]^T \in \mathbb{R}^{2m} \quad (2.5)$$

The following definition will be useful.

DEFINITION 2.1 The matrix $\mathcal{X}^T \mathcal{X}$ is defined as the **confidence matrix** associated with the harmonic regressor x in (2.4). ■

The name “confluence matrix” has been chosen to reflect the fact that for overparametrized regressors $x \in \mathbb{R}^N$, $N > 2m$, the N signal channels of the regressor are effectively combined into a smaller number of $2m$ channels by properties of this matrix. The confluence matrix will play an important role in the analysis which follows.

Equations (2.1)-(2.5) taken together will be referred to as a *harmonic adaptive system*. Collectively, these equations define an important open-loop mapping from the error signal e to the estimated output \hat{y} . Because of its importance, this mapping will be denoted by the special character \mathcal{H} , i.e.,

$$\hat{y} = \mathcal{H}[e] \quad (2.6)$$

The special structure of \mathcal{H} is depicted in Figure 2.1.

REMARK 2.1 The definition of $\Gamma(s)$ is left intentionally general to include analysis of the gradient algorithm (i.e., with the choice $\Gamma(s) = 1/s$), the gradient algorithm with leakage (i.e., $\Gamma(s) = 1/(s + \sigma)$; $\sigma \geq 0$), proportional-plus-integral adaptation (i.e., $\Gamma(s) = k_p + k_i/s$), or arbitrary linear adaptation algorithms of the designer's choosing. Adaptation laws which

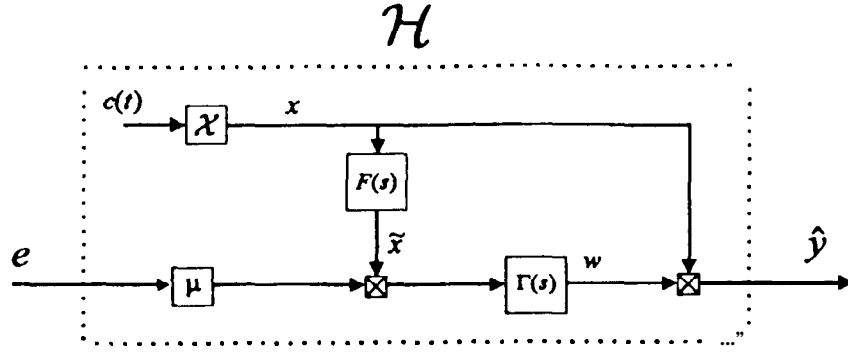


Figure 2.1: LTV operator $\hat{y} = \mathcal{H}[e]$ for adaptive system with harmonic regressor z , adaptation law $\Gamma(s)$, and regressor filter $F(s)$

are nonlinear or normalized (e.g., divided by the norm of the regressor), are not considered here since they do not have an equivalent LTI representation $I'(p)$. ■

REMARK 2.2 The use of the regressor filter $F(s)$ in (2.3) allows the unified treatment of many important adaptation algorithms including the well-known Filtered-X algorithm from the signal processing literature [8][5][2][9], and the Augmented Error algorithm of Monopoli [4]. Since z is comprised purely of sinusoidal components and F in (2.3) is, stable, all subsequent analysis will assume that the filter output \tilde{x} has reached a steady-state condition. ■

The following result taken from [1] will be needed which gives necessary and sufficient conditions for the operator \mathcal{H} to be LTI.

THEOREM 2.1 (LTI Representation Theorem) *Let the regressor $x(t)$ in the adaptive system (1)-(9) be given by the general multitone harmonic expression (24) where the frequencies $\{\omega_i\}_{i=1}^m$ are distinct, nonzero, and $|F(j\omega_i)| > 0$ for all i .*

Then,

(i) *The mapping \mathcal{H} from e to \hat{y} is exactly representable as the linear time-invariant operator,*

$$\mathcal{H}: \quad \hat{y} = \bar{H}(p)e \quad (2.7)$$

if and only if the matrix X in (0??4) satisfies the following X -Orthogonality(XO) condition,

X -Orthogonality(XO) Condition:

$$X^T X = D^2 \quad (2.8)$$

$$D^2 \triangleq \begin{bmatrix} d_1^2 \cdot I_{2 \times 2} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & d_m^2 \cdot I_{2 \times 2} \end{bmatrix} \in R^{2m \times 2m} \quad (2.9)$$

where, $d_i^2 \geq 0, i = 1, \dots, m$ are scalars and $I_{2 \times 2} \in R^{2 \times 2}$ is a matrix identity.

(ii) $\bar{H}(s)$ in (5.7) is given in closed-form as,

$$\bar{H}(s) = \mu \sum_{i=1}^m d_i^2 \cdot H_i(s) \quad (2.10)$$

$$H_i(s) = \frac{F_R(i)}{2} \left(\Gamma(s - j\omega_i) + \Gamma(s + j\omega_i) \right) + \frac{F_I(i)}{2j} \left(\Gamma(s - j\omega_i) - \Gamma(s + j\omega_i) \right) \quad (2.11)$$

$$F_R(i) \triangleq \text{Re}(F(j\omega_i)); F_I(i) \triangleq \text{Im}(F(j\omega_i)) \quad (2.12)$$

■

Simply stated, the XO condition in Theorem 2.1 implies that a harmonic adaptive system is LTI if and only if its confluence matrix (defined in 3.1) is pairwise diagonal.

The following result taken from [1] shows that in the general case where the XO condition is not satisfied, the mapping 'H can always be decomposed into a *parallel connection* of an LTI subsystem and an LTV perturbation.

THEOREM 2.2 (LTI/LTV Decomposition) Consider the adaptive system (2.1)-(2.3) with harmonic regressor (224)(%5). Then,

(i) In general the mapping \mathcal{H} from e to \hat{y} can be expressed as the parallel connection of an LTI block $\bar{H}(s)$, and an LTV perturbation block $\tilde{\Delta}$,

$$\mathcal{H}: \quad \hat{y} = \bar{H}(p)e + \tilde{\Delta}[e] \quad (2.13)$$

where,

$$\bar{H}(s) \triangleq \mu \sum_{i=1}^m d_i^2 \cdot H_i(s) \quad (2.14)$$

$$\tilde{\Delta}[e] \triangleq \mu c(t)^T \Delta \Gamma(p) [\mathcal{F}c(t)e] \quad (2.15)$$

$$\Delta \triangleq \mathcal{X}^T \mathcal{X} - D^2 \quad (2.16)$$

$$F \triangleq \text{blockdiag}\{\mathcal{F}_i\} \in R^{2m \times 2m} \quad (2.17)$$

$$\mathcal{F}_i \triangleq \begin{bmatrix} F_R(i) & F_I(i) \\ -F_I(i) & F_R(i) \end{bmatrix} \in \mathbb{R}^{2 \times 2}; \quad \text{for } i = 1, \dots, m \quad (2.18)$$

$$F_R(i) \triangleq \text{Re}(F(j\omega_i)); \quad F_I(i) \triangleq \text{Im}(F(j\omega_i)) \quad (2.19)$$

and where $H_i(s)$ is as defined in (2.11) of Theorem 2.1, and D is chosen (non-uniquely) as any matrix of the 2×2 block-diagonal form (S.9).

(ii) If the adaptation law $I(s)$ is stable with infinity norm $\|\Gamma(s)\|_\infty$, then the gain of the LTV perturbation can be bounded from above as,

$$\|\tilde{\Delta}\|_{2i} \leq \mu m \bar{\sigma}(\Delta) \|\Gamma(s)\|_\infty \max_i |F(\omega_i)| \quad (2.20)$$

where $\|\cdot\|_{2i}$ denotes the induced \mathcal{L}_2 -norm of the indicated operator.

REMARK 2.3 The LTI/LTV decomposition in Theorem 2.2 is important for adaptive systems which do not exactly satisfy the XO condition. In this case, the adaptive system can be analyzed using modern robust control methods (i.e., small gain theorem) making use of the analytic expression (2.14) for the LTI block $\bar{H}(s)$ and the norm bound (2.20) on the time-varying perturbation block $\tilde{\Delta}$ [7][11]. ■

REMARK 2.4 The need for $\|\Gamma(s)\|_\infty$ to exist in Theorem 2.2 (part ii) requires that the adaptive law use some type of “leakage” (cf., Ioannou and Kokotovic [?]). The possibility of less conservative norm-bound remains as an open issue. ■

3 TAP DELAY-LINE (TDL) BASIS

3.1 Single Tone Case

In 1977, Glover [3] made the interesting discovery that an LTI system arises if the regressor x of an adaptive feedforward system is defined by filtering a sinusoid through a long tap delay line (TDL). Glover’s result can be understood simply in terms of the XO condition of Theorem 2.1.

Let the regressor $x(t) = [x_1(t), \dots, x_N(t)]^T \in \mathbb{R}^N$ be defined by filtering a single frequency $\omega_1 > 0$,

$$\xi(t) = a_{11} \sin(\omega_1 t) + a_{12} \cos(\omega_1 t) \quad (3.1)$$

through a TDL with N taps and tap delay T , i.e.,

$$x_\ell(t) = e^{-(\ell-1)pT} \xi(t), \quad \ell = 1, \dots, N \quad (3.2)$$

where the term $e^{-(\ell-1)pT}$ in the differential operator p represents a delay of $(\ell-1)T$ time units. If $x(t)$ is written in the form $x = \mathcal{X}c(t)$, then it can be shown (i.e., set $i = 1$ and

$m = 1$ in Theorem A.1 of Appendix A), that the confluence matrix is given by,

$$\mathbf{X}^T \mathbf{X} = \frac{N}{2} \alpha_1^2 \cdot I_{2 \times 2} + \frac{1}{2} \mathcal{B}_N(\omega_1 T) A_1^T R_{11} A_1 \quad (3.3)$$

where,

$$\alpha_1^2 = a_{11}^2 + a_{12}^2 \quad (3.4)$$

$$A_1 = \begin{bmatrix} a_{12} & -a_{11} \\ a_{11} & a_{12} \end{bmatrix}; \quad R_{11} = \begin{bmatrix} -\cos(N-1)\omega_1 T & \sin(N-1)\omega_1 T \\ \sin(N-1)\omega_1 T & \cos(N-1)\omega_1 T \end{bmatrix} \quad (3.5)$$

$$\mathcal{B}_N(\nu) \triangleq \frac{\sin N\nu}{\sin \nu} \quad (3.6)$$

The first term of (3.3) (a pairwise diagonal matrix), increases as N , while the second term remains bounded, Normalizing the adaptation gain to $\mu = \bar{\mu}/N$ for some $\bar{\mu} > 0$ (to prevent unbounded feedforward gain), and taking the limit as $N \rightarrow \infty$ gives,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathcal{X}^T \mathcal{X} = \frac{1}{2} \alpha_1^2 \cdot I_{2 \times 2} \quad (3.7)$$

Since the XO condition Theorem 2.1 is satisfied asymptotically as N gets large, the system becomes asymptotically LTI and is given as,

$$\mathcal{H}: \quad \hat{y} = \frac{\bar{\mu}}{2} \alpha_1^2 \cdot H_1(p) e \quad (3.8)$$

where $H_1(s)$ is given by (2.11)(2.12), and where $\alpha_1^2 = a_{11}^2 + a_{12}^2$. Restricting the choice of filter to $F(s) = 1$ in (3.8), and the choice of adaptation algorithm to the gradient algorithm (i.e., $\Gamma(s) = 1/s$), gives precisely Glover's result (proved originally in the discrete-time case [3]). This result is recovered here as a special case of Theorem 2.1.

3.2 Multi-Tone Case

Rigorously speaking, Glover's LTI analysis of TDL regressors is only applicable in the limiting case when $N = \infty$, i.e., it ignores the contribution of the LTV subsystem for finite N , and does not provide any characterization of the convergence properties as N increases. In contrast, a more complete solution can be found by putting Glover's results into a modern robust control setting. This will be done in the present section by applying the LTI/LTV decomposition of Theorem 2.2 to the TDL regressor case. First, a definition will be useful.

DEFINITION 3.1 Given time delay T and spacing parameter $0 < \underline{\nu} < \pi/2$, a Bounded Tone Set $\Omega(m, T, \underline{\nu})$ is defined as any set of m frequencies $\{\omega_i\}_{i=1}^m$, such that,

$$\Omega(m, T, \underline{\nu}) \triangleq \left\{ \{\omega_i\}_{i=1}^m : \quad 0 < \underline{\nu} < \pi/2; \right.$$

$$0 < \underline{\nu} < \omega_i T \leq \pi - \underline{\nu} \text{ for all } i = 1, \dots, m;$$

$$|\omega_i - \omega_j|T \geq 2\underline{\nu} \quad \text{for all } i \neq j \} \quad (3.9)$$

■

Simply stated, a Bounded Tone Set is a set of frequencies $\{\omega_i\}_{i=1}^m$ which are bounded away from 0, π/T and each other. The definition is not very restrictive since any signal comprised of a finite number of distinct sinusoids lies in a Bounded Tone Set when T is chosen sufficiently small (i.e., to ensure Nyquist sampling of its highest component). Definition 3.1 is conveniently used to define the minimum *spacing* parameter $\underline{\nu}$ which will play an essential role in many subsequent results.

The main result of the paper follows.

THEOREM 3.1 (Tap Delay-Line Basis) *Consider the adaptive system (2.1)-(2.9) with harmonic regressor (2.4)(2.5), and input/output mapping \mathcal{H} in (2.6). Let* the component of the regressor $x = [x_1, \dots, x_N]^T \in \mathbb{R}^N$ be defined by filtering a signal $\xi(t) \in \mathbb{R}^1$ through a tap delay line with N taps and tap delay T , t. e.,*

$$x_\ell = e^{-(\ell-1)T} \xi, \ell = 1, \dots, N \quad (3.10)$$

where the measured signal ξ is given by the following sum of m sinusoids,

$$\xi(t) = \sum_{i=1}^m \alpha_i \sin(\omega_i t + \phi_i); \alpha_i > 0 \quad (3.11)$$

and frequencies $\{\omega_i\}_{i=1}^m$ lie in a bounded tone set $\Omega(m, T, \underline{\nu})$.

Then,

(i) The regressor $z(t)$ can be written in harmonic form (2.4)(2.5) where the matrix $X \in \mathbb{R}^{N \times 2m}$ satisfies,

$$X^T X = D^2 + \Delta \quad (3.12)$$

$$D^2 = \frac{N}{2} \begin{bmatrix} \alpha_1^2 \cdot I_{2 \times 2} & 0 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \alpha_m^2 \cdot I_{2 \times 2} \end{bmatrix} \in \mathbb{R}^{2m \times 2m} \quad (3.13)$$

and the matrix perturbation $\Delta \triangleq X^T X - D^2$ is norm-bounded as,

$$\bar{\sigma}(\Delta) \leq \frac{m\pi\alpha_{\max}^2}{2\underline{\nu}}; \quad \alpha_{\max} \triangleq \max_i \{\alpha_i\} \quad (3.14)$$

(ii) (**LTI/LTV Decomposition**)

The mapping \mathcal{H} from $\hat{\mathbf{y}}$ to e can be uniquely deco-reposed into the parallel connection of an LTI block $\bar{H}(s)$, and an LTV perturbation block \mathbf{A} ,

$$\mathcal{H} : \quad \hat{\mathbf{y}} = \bar{H}(p)e + \tilde{\Delta}[e] \quad (3.15)$$

where,

$$\bar{H}(s) \triangleq \mu \frac{N}{2} \sum_{i=1}^m \alpha_i^2 \cdot H_i(s) \quad (3.16)$$

$$\tilde{\Delta}[e] \triangleq \mu c(t)^T \Delta \Gamma(p) [\mathcal{F}c(t)e] \quad (3.17)$$

where the perturbation matrix \mathbf{A} is defined in (9.12) with norm bound (3.14), and $H_i(s)$ is given by (2.11) of Theorem ??1.

Furthermore, if the adaptation law $\Gamma(s)$ is stable with infinity norm $\|\Gamma(s)\|_\infty$, then the gain of the LTV perturbation can be bounded as,

$$\|\tilde{\Delta}\|_{2i} \leq \frac{\mu m^2 \pi}{2\nu} \cdot \left(\alpha_{\max}^2 \|\Gamma(s)\|_\infty \max_i |F(\omega_i)| \right) \quad (3.18)$$

where $\|\cdot\|_{2i}$ indicates the induced \mathcal{L}_2 -norm.

(iii) (**Normalized Adaptation Gain**)

By normalizing the adaptive gain to $\mu = \bar{\mu}/N$, the operators in (9.16)(3.17) of (ii) become,

$$\bar{H}(s) = \frac{\bar{\mu}}{2} \sum_{i=1}^m \alpha_i^2 \cdot H_i(s) \quad (3.19)$$

$$\tilde{\Delta}[e] \triangleq \frac{\bar{\mu}}{N} c(t)^T \Delta \Gamma(p) [\mathcal{F}c(t)e] \quad (3.20)$$

and the upper bound on the gain of the LTV perturbation in (9.18) becomes,

$$\|\tilde{\Delta}\|_{2i} \leq \frac{\bar{\mu} m^2 \pi}{2N\nu} \cdot \left(\alpha_{\max}^2 \|\Gamma(s)\|_\infty \max_i |F(\omega_i)| \right) \quad (3.21)$$

where $\|\Gamma(s)\|_\infty$ is resumed to exist.

(iv) (**Asymptotic Properties**)

If the adaptation law $\Gamma(s)$ is stable (with bounded infinity norm $\|\Gamma(s)\|_\infty$), and the adaptation gain is normalized as $\mu = \bar{\mu}/N$ for $\bar{\mu} > 0$ constant, then as $N \rightarrow \infty$ the mapping \mathcal{H} becomes LTI with asymptotic transfer function,

$$\bar{H}(s) = \frac{\bar{\mu}}{2} \sum_{i=1}^m \alpha_i^2 \cdot H_i(s) \quad (3.22)$$

PROOF:

Proof of (i): From Theorem A.1 the confluence matrix is given by $\mathcal{X}^T \mathcal{X} = \mathcal{M}$ where M is given by (A.52)-(A.55). Hence, $A = M - D^2 \in \mathbb{R}^{2m \times 2m}$ has the symmetric block 2-by-2 structure (A.77) used in Lemma A.5. Applying the result (A.78) of Lemma A.5 gives,

$$\bar{\sigma}(\Delta) \leq m \cdot \max_{i,j} \bar{\sigma}(\Delta_{ij}) \leq m \cdot \max_{i,j} \frac{\pi \alpha_i \alpha_j}{2\underline{\nu}} \quad (3.23)$$

$$\leq \frac{m\pi}{2\underline{\nu}} \max_i \{\alpha_i^2\} \triangleq \frac{m\pi}{2\underline{\nu}} \alpha_{\max}^2 \quad (3.24)$$

where use has been made of property P.5 of Lemma A.4 in equation (3.23).

Proof of (ii): Results follow by applying the LTI/LTV Decomposition of Theorem 2.2 noting from (3.13) that in the present case $d_i^2 = \frac{N}{2} \alpha_i^2$.

Proof of (iii): Simply substitute $\mu = \bar{\mu}/N$ into the LTI and LTV blocks, where $\bar{\mu} > 0$ is a constant.

Proof of (iv): It is seen that the normalized LTI transfer function $\bar{H}(s)$ in (3.19) remains unaffected as N increases while the normalized LTV perturbation in (3.20) goes to zero as N increases. Hence, as $N \rightarrow \infty$ the mapping \mathcal{H} becomes LTI with asymptotic transfer function given in (3.22), as desired. ■

For convenience, the results of Theorem 3.1 are summarized in Figure 3.1. Specifically, Figure 3.1 Part a. shows the harmonic adaptive system with TDL basis and normalized adaptation gain $\mu = \bar{\mu}/N$; Part b. shows the equivalent decomposition into an LTI block and a norm bounded LTV perturbation block. Note that the time-varying perturbation block goes to zero asymptotically as N becomes large.

REMARK 3.1 The asymptotic result (iv) of Theorem 3.1 follows essentially from the special form of the confluence matrix $\mathcal{X}^T \mathcal{X}$ in (3.12) which arises in the TDL case. Specifically, in relation (3.12), the matrix D^2 given by (3.13) (and hence the associated LTI block) *grows linearly* with the number of taps N , while the perturbation matrix A (and hence the associated LTV block) *remains bounded* as N increases.

Hence, when the LTI and LTV paths are normalized by $1/N$ through choice of adaptation gain $\mu = \bar{\mu}/N$ (as shown in Figure 3.1) and the limit is taken as N becomes large, the LTI part remains constant while the norm bound on the LTV part decreases as $1/N$. This indicates that the LTV part can be made arbitrarily small by choosing N sufficiently large, while the LTI part remains unaffected. ■

REMARK 3.2 Interestingly, the bound on the LTV perturbation A in (3.21) depends on the boundedness of the tone-set through the minimum spacing parameter $\underline{\nu} > 0$ defined in Definition 3.1. Specifically, a smaller $\underline{\nu}$ requires a larger N to justify the asymptotic

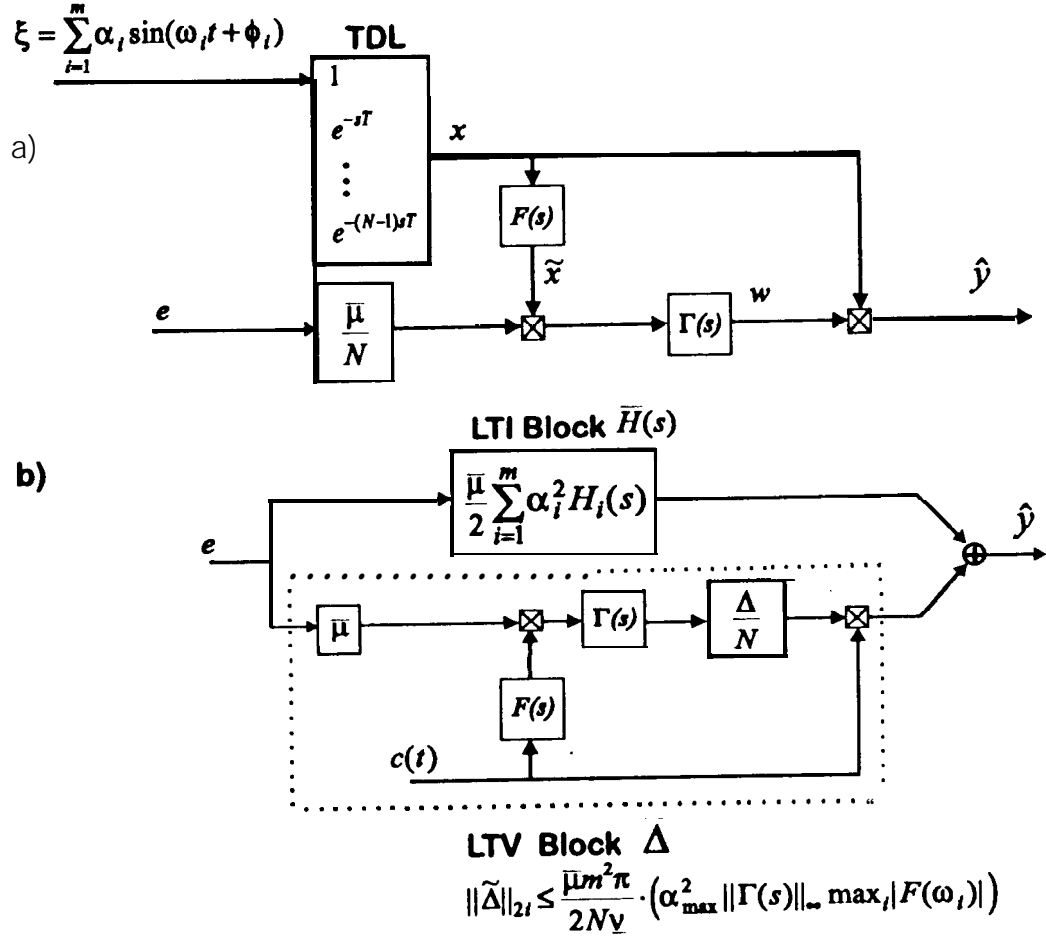


Figure 3.1: LTI/LTV decomposition of \mathcal{H} for harmonic adaptive system with TDL basis

approximation to the same degree. This relationship precisely characterizes the effect of tone spacing on the convergence of the LTV subsystem, and settles a long standing question on the role of tone spacing in determining the asymptotic properties of adaptive systems with long tap delay lines. It is worth noting that these results are consistent with the heuristic discussions of tone spacing found Glover's original paper [3] (cf., Section IV, pp. 488).

Also interesting is the appearance of m^2 in the numerator of the norm bound (3.21). This indicates that if the number of tones m in the regressor is increased, one must increase N as the *square* of m to justify the asymptotic approximation to the same degree. This dependence on the *tone* count also appears to be new. ■

4 CONCLUSIONS

It has been shown that adaptive systems whose regressors are formed by filtering multitone sinusoidal signals through *tap* delay-lines satisfy the XO condition (and hence have LTI representations) in the limit as the number of taps becomes infinite. This result is of significant practical importance since tap delay lines, FIR filter representations, etc., are commonly used in many adaptive signal processing and communications applications,

For the more realistic case of a finite number of taps, Theorem 3.1 extends Glover's work by putting the problem into a modern robust control setting. Specifically, the unwanted time-varying terms (arising in Glover's expansion of the adaptive operator), are characterized precisely in terms of a norm bounded perturbation (3.21). This permits a rigorous treatment of the time-varying dynamics using modern robust control theory, and exposes the exact nature of the convergence to an LTI system as the number of taps is increased. The norm bound (3.21) is seen to be proportional to $m^2/(N\underline{\nu})$ which clearly indicates the size of the LTV perturbation as a function of the number of taps N , the minimum tone spacing parameter $\underline{\nu}$, and the number of tones m . To the author's knowledge, this is the first time these dependencies have been made explicit. Using this new model, the choice of N can be determined from precise H_∞ conditions rather than the heuristic "rules of thumb" given in Glover's original paper, and precise statements can be made about the stability of the system even in the presence of the LTV subsystem.

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A APPENDIX: Properties of TDL Regressors

The purpose of Appendix A is to provide the detailed structure of the confluence matrix $\mathcal{X}^T \mathcal{X}$ for a tap delay-line basis (i.e., in Theorem A.1), and additional supporting results which are needed to prove Theorem 3.1.

The following definitions will be used throughout Appendix A:

$$C_i \triangleq \begin{bmatrix} 1 \\ \cos \omega_i T \\ \vdots \\ \cos(N-1)\omega_i T \end{bmatrix} \in R^N; \quad S_i \triangleq \begin{bmatrix} 0 \\ \sin \omega_i T \\ \vdots \\ \sin(N-1)\omega_i T \end{bmatrix} \in R^N \quad (\text{A.1})$$

$$\underline{c}_{ij} \triangleq \cos((N-1)(\omega_i - \omega_j)T/2) \quad (\text{A.2})$$

$$\underline{s}_{ij} \triangleq \sin((N-1)(\omega_i - \omega_j)T/2) \quad (\text{A.3})$$

$$c_{ij} \triangleq \cos((N-1)(\omega_i + \omega_j)T/2) \quad (\text{A.4})$$

$$s_{ij} \triangleq \sin((N-1)(\omega_i + \omega_j)T/2) \quad (\text{A.5})$$

$$\underline{R}_{ij} \triangleq \begin{bmatrix} \underline{c}_{ij} & \underline{s}_{ij} \\ -\underline{s}_{ij} & \underline{c}_{ij} \end{bmatrix} \quad (\text{A.6})$$

$$R_{ij} \triangleq \begin{bmatrix} -c_{ij} & s_{ij} \\ s_{ij} & c_{ij} \end{bmatrix} \quad (\text{A.7})$$

$$\xi(t) \triangleq \sum_{i=1}^m \alpha_i \sin(\omega_i t + \phi_i) = \sum_{i=1}^m a_{i1} \sin(\omega_i t) + a_{i2} \cos(\omega_i t) \quad (\text{A.8})$$

$$A_i \triangleq \begin{bmatrix} a_{i2} & -a_{i1} \\ a_{i1} & a_{i2} \end{bmatrix} \quad (\text{A.9})$$

$$\mathcal{B}_N(\nu) \triangleq \frac{\sin N\nu}{\sin \nu} \quad (\text{A.10})$$

LEMMA A.1 *Let $\mathcal{B}_N(\nu)$ be defined by (A.10). Then, on the interval $0 \leq \nu \leq \pi$, the following inequality holds,*

$$|\mathcal{B}_N(\nu)| \leq \frac{\pi}{2\tau(\nu)}; \quad \text{for } 0 \leq \nu \leq \pi \quad (\text{A.11})$$

where,

$$\tau(\nu) \triangleq \min(\nu, \pi - \nu) \quad (\text{A.12})$$

PROOF: A sinusoid $\sin \nu$ can be bounded below on the interval $0 \leq \nu \leq \pi$ by piecewise linear segments as follows,

$$|\sin \nu| \geq \frac{1}{\pi} \min(2\nu, 2(\pi - \nu)); \text{ for } 0 \leq \nu \leq \pi \quad (\text{A.13})$$

Hence,

$$\frac{\sin N\nu}{\sin \nu} \leq \frac{\pi |\sin N\nu|}{\min(2\nu, 2(\pi - \nu))} \leq \frac{\pi}{2 \cdot \tau(\nu)} \quad (\text{A.14})$$

■

LEMMA A.2 Let $\mathcal{B}_N(\nu)$ be defined by (A.10). Then for frequencies $\{\omega_i\}_{i=1}^m$ in a bounded tone set $\Omega(m, T, \underline{\nu})$, the following inequalities hold,

$$|\mathcal{B}_N(\omega_i T)| \leq \frac{\pi}{2\underline{\nu}} \quad (\text{A.15})$$

$$|\mathcal{B}_N((\omega_i - \omega_j)T/2)| \leq \frac{\pi}{2\underline{\nu}} \quad (\text{A.16})$$

$$|\mathcal{B}_N((\omega_i + \omega_j)T/2)| \leq \frac{\pi}{2\underline{\nu}} \quad (\text{A.17})$$

PROOF: By properties of the Bounded Tone Set (3.9), the following inequalities can be shown to hold for any $\omega_i, \omega_j \in \Omega(m, T, \underline{\nu})$,

$$2\underline{\nu} \leq |\omega_i - \omega_j|T \leq \pi - 2\underline{\nu} \quad (\text{A.18})$$

$$2\underline{\nu} \leq |\omega_i + \omega_j|T \leq 2\pi - 2\underline{\nu} \quad (\text{A.19})$$

Proof of (A.15):

$$\mathcal{B}_N(\omega_i T) \leq \frac{\pi}{2\tau(\omega_i T)} = \frac{\pi}{2\min(\omega_i T, \pi - \omega_i T)} \leq \frac{\pi}{2\underline{\nu}} \quad (\text{A.20})$$

Here, the first inequality in (A.20) follows by (All) of Lemma A1; the second equality follows by definition of τ in (A.12); and the last inequality follows by properties of the bounded tone set $\Omega(m, T, \underline{\nu})$ in (3.9).

Proof of (A.16):

$$\mathcal{B}_N((\omega_i - \omega_j)T/2) \leq \frac{\pi}{2\tau((\omega_i - \omega_j)T/2)} \quad (\text{A.21})$$

$$= \frac{\pi}{\min(|\omega_i - \omega_j|T, 2\pi - |\omega_i - \omega_j|T)} \quad (\text{A.22})$$

$$\leq \frac{\pi}{\min(2\underline{\nu}, 2\pi - (\pi - 2\underline{\nu}))} = \frac{\pi}{2\underline{\nu}} \quad (\text{A.23})$$

Here, equation (A.21) follows by (All); (A.22) follows by the definition of $\tau(\cdot)$ in (A.12) and the fact that the function $\mathcal{B}_N(\cdot)$ is even; and (A.23) follows from (A.18).

Proof of (A.17):

$$\mathcal{B}_N((\omega_i + \omega_j)T/2) \leq \frac{\pi}{2\tau((\omega_i + \omega_j)T/2)} \quad (\text{A.24})$$

$$= \frac{\pi}{\min(|\omega_i + \omega_j|T, 2\pi - |\omega_i + \omega_j|T)} \quad (\text{A.25})$$

$$\leq \frac{\pi}{\min(2\nu, 2\pi - (2\pi - 2\nu))} = \frac{\pi}{2\nu} \quad (\text{A.26})$$

Here, equation (A.24) follows by (All); (A.25) follows by the definition of $\tau(\cdot)$ in (A.12) and the fact that the function $\mathcal{B}_N(\cdot)$ is even; and (A.26) follows from (A.19). ■

LEMMA A.3 *Let C_i, S_i be as defined in (A.1). Then the following identities hold,*

$$S_i^T S_j = \frac{1}{2} (\underline{c}_{ij} \cdot \mathcal{B}_N((\omega_i - \omega_j)T/2) - c_{ij} \cdot \mathcal{B}_N((\omega_i + \omega_j)T/2)) \quad (\text{A.27})$$

$$C_i^T C_j = \frac{1}{2} (\underline{c}_{ij} \cdot \mathcal{B}_N((\omega_i - \omega_j)T/2) + c_{ij} \cdot \mathcal{B}_N((\omega_i + \omega_j)T/2)) \quad (\text{A.28})$$

$$C_j^T S_i = S_i^T C_j = \frac{1}{2} (\underline{s}_{ij} \cdot \mathcal{B}_N((\omega_i + \omega_j)T/2) + \underline{c}_{ij} \cdot \mathcal{B}_N((\omega_i - \omega_j)T/2)) \quad (\text{A.29})$$

$$C_i^T S_j = S_j^T C_i = \frac{1}{2} (\underline{s}_{ij} \cdot \mathcal{B}_N((\omega_i + \omega_j)T/2) - \underline{c}_{ij} \cdot \mathcal{B}_N((\omega_i - \omega_j)T/2)) \quad (\text{A.30})$$

where $\underline{c}_{ij}, \underline{s}_{ij}, s_{ij}, c_{ij}, \mathcal{B}_N$ are as defined in (A.2)-(A.5), and (A.10).

PROOF: Use will be made of the following identity,

$$\sum_{\ell=1}^N e^{j(\ell-1)\nu} = \frac{1 - e^{jN\nu}}{1 - e^{j\nu}} \quad (\text{A.31})$$

$$= e^{j(N-1)\nu/2} \cdot \frac{\sin(N\nu/2)}{\sin(\nu/2)} \triangleq e^{j(N-1)\nu/2} \cdot \mathcal{B}_N(\nu/2) \quad (\text{A.32})$$

Proof of (A.27):

$$S_i^T S_j = \sum_{\ell=1}^N \sin(\omega_i(\ell-1)T) \sin(\omega_j(\ell-1)T) \quad (\text{A.33})$$

$$= \frac{1}{2} \sum_{\ell=1}^N \cos((\omega_i - \omega_j)(\ell-1)T) - \cos((\omega_i + \omega_j)(\ell-1)T) \quad (\text{A.34})$$

$$= \frac{1}{2} \mathbf{Re} \left\{ \sum_{\ell=1}^N e^{j(\omega_i - \omega_j)(\ell-1)T} - e^{j(\omega_i + \omega_j)(\ell-1)T} \right\} \quad (\text{A.35})$$

$$= \frac{1}{2} \mathbf{Re} \left\{ e^{j(N-1)(\omega_i - \omega_j)T/2} \mathcal{B}_N((\omega_i - \omega_j)T/2) \right. \\ \left. - e^{j(N-1)(\omega_i + \omega_j)T} \mathcal{B}_N((\omega_i + \omega_j)T/2) \right\} \quad (\text{A.36})$$

$$= \frac{1}{2} \left(\underline{c}_{ij} \mathcal{B}_N((\omega_i - \omega_j)T/2) - c_{ij} \mathcal{B}_N((\omega_i + \omega_j)T/2) \right) \quad (\text{A.37})$$

Here, $\mathbf{Re}(\cdot)$ denotes the real part of the indicated expression. Equations (A.33)-(A.35) follow by standard trig formulas; equation (A.36) follows by using identity (A.32); and (A.37) follows by the definition of s_{ij} and \underline{s}_{ij} in (A.2)-(A.5).

Proof of (A.28):

$$C_i^T C_j = \sum_{\ell=1}^N \cos(\omega_i(\ell-1)T) \cos(\omega_j(\ell-1)T) \quad (\text{A.38})$$

$$= \frac{1}{2} \sum_{\ell=1}^N \cos((\omega_i - \omega_j)(\ell-1)T) + \cos((\omega_i + \omega_j)(\ell-1)T) \quad (\text{A.39})$$

$$= \frac{1}{2} \mathbf{Re} \left\{ \sum_{\ell=1}^N e^{j(\omega_i - \omega_j)(\ell-1)T} + e^{j(\omega_i + \omega_j)(\ell-1)T} \right\} \quad (\text{A.40})$$

$$= \frac{1}{2} \mathbf{Re} \left\{ e^{j(N-1)(\omega_i - \omega_j)T/2} \mathcal{B}_N((\omega_i - \omega_j)T/2) \right. \quad (\text{A.41})$$

$$\left. + e^{j(N-1)(\omega_i + \omega_j)T} \mathcal{B}_N((\omega_i + \omega_j)T/2) \right\} \quad (\text{A.42})$$

$$= \frac{1}{2} \left(\underline{c}_{ij} \mathcal{B}_N((\omega_i - \omega_j)T/2) + c_{ij} \mathcal{B}_N((\omega_i + \omega_j)T/2) \right) \quad (\text{A.43})$$

Here, equations (A.38)-(A.40) follow by standard trig formulas; equation (A.42) follows by using identity (A.32); and (A.43) follows by the definition of c_{ij} and \underline{c}_{ij} in (A.2)-(A.5).

Proof of (A.29):

$$C_j^T S_i = S_i^T C_j = \sum_{\ell=1}^N \sin(\omega_i(\ell-1)T) \cos(\omega_j(\ell-1)T) \quad (\text{A.44})$$

$$= \frac{1}{2} \sum_{\ell=1}^N \sin((\omega_i + \omega_j)(\ell-1)T) + \sin((\omega_i - \omega_j)(\ell-1)T) \quad (\text{A.45})$$

$$= \frac{1}{2} \text{Im} \left\{ \sum_{\ell=1}^N e^{j(\omega_i + \omega_j)(\ell-1)T} + e^{j(\omega_i - \omega_j)(\ell-1)T} \right\} \quad (\text{A.46})$$

$$= \frac{1}{2} \text{Im} \left\{ e^{j(N-1)(\omega_i + \omega_j)T/2} \mathcal{B}_N((\omega_i + \omega_j)T/2) \right. \\ \left. + e^{j(N-1)(\omega_i - \omega_j)T/2} \mathcal{B}_N((\omega_i - \omega_j)T/2) \right\} \quad (\text{A.47})$$

$$= \frac{1}{2} (s_{ij} \mathcal{B}_N((\omega_i + \omega_j)T/2) + \underline{s}_{ij} \mathcal{B}_N((\omega_i - \omega_j)T/2)) \quad (\text{A.48})$$

Here, $\text{Im}(\cdot)$ denotes the imaginary part of the indicated expression. Equations (A.44)-(A.46) follow by standard trig formulas; equation (A.47) follows by using identity (A.32); and (A.48) follows by the definition of s_{ij} and \underline{s}_{ij} in (A.2)-(A.5).

Proof of (A.30): This relation follows by reversing the roles of i and j in the proof of (A.29), making use of the antisymmetric property of $\sin(-\theta) = -\sin(\theta)$ and symmetric property of $\mathcal{B}_N(-\nu) = \mathcal{B}_N(\nu)$. ■

THEOREM A.1 (Confluence Matrix for a TDL) *Let the components of the regressor $\mathbf{x} = [x_1, \dots, x_N]^T \in \mathbb{R}^N$ be defined by filtering a signal $\xi(t) \in \mathbb{R}^1$ through a tap delay line with N taps and tap delay T , i.e.,*

$$x_\ell = e^{-(\ell-1)sT} \xi, \ell = 1, \dots, N \quad (\text{A.49})$$

where the measured signal ξ is given by the following sum of m distinct sinusoids,

$$\xi(t) = \sum_{i=1}^m \alpha_i \sin(\omega_i t + \phi_i) = \sum_{i=1}^m a_{i1} \sin(\omega_i t) + a_{i2} \cos(\omega_i t) \quad (\text{A.50})$$

Then, the regressor $\mathbf{z}(t)$ is of the harmonic form,

$$\mathbf{x} = \mathbf{X} \mathbf{c}(t) \quad (\text{A.51})$$

where $\mathbf{X} \in \mathbb{R}^{N \times 2m}$, $\mathbf{c}(t)$ is defined in (2.5), and its confluence matrix is given by,

$$\mathbf{X}^T \mathbf{X} = \begin{bmatrix} \mathcal{M}_{11} & \dots & \mathcal{M}_{1m} \\ \vdots & \ddots & \vdots \\ \mathcal{M}_{m1} & \dots & \mathcal{M}_{mm} \end{bmatrix} \in \mathbb{R}^{2m \times 2m} \quad (\text{A.52})$$

$$\mathcal{M}_{ij} \in \mathbb{R}^{2 \times 2}; i = 1, \dots, m; j = 1, \dots, m \quad (\text{A.53})$$

$$M_{ij} = \frac{1}{2} A_i^T \left(\mathcal{B}_N((\omega_i - \omega_j)T/2) R_{ij} + \mathcal{B}_N((\omega_i + \omega_j)T/2) R_{ij} \right) A_j \quad (\text{A.54})$$

$$M_{ii} = \frac{N}{2} \alpha_i^2 \cdot I_{2 \times 2} + \frac{1}{2} \mathcal{B}_N(\omega_i T) A_i^T R_{ii} A_i \quad (\text{A.55})$$

where definitions (A. 1)-(A. 10) have been used.

PROOF: Using standard trigonometric identities, the ℓ th element x_ℓ of the delayed regressor (A.49) can be expanded as,

$$x_\ell(t) = e^{-(\ell-1)sT} \xi = \sum_{i=1}^m a_{i1} \sin(\omega_i(t - (\ell-1)T)) + a_{i2} \cos(\omega_i(t - (\ell-1)T)) \quad (\text{A.56})$$

$$\begin{aligned} &= \sum_{i=1}^m a_{i1} \left(\cos(\omega_i(\ell-1)T) \sin \omega_i t - \sin(\omega_i(\ell-1)T) \cos \omega_i t \right) \\ &+ a_{i2} \left(\sin(\omega_i(\ell-1)T) \sin \omega_i t + \cos(\omega_i(\ell-1)T) \cos \omega_i t \right) \end{aligned} \quad (\text{A.57})$$

Using (A.57), the full regressor $x(t)$ can be decomposed in terms of the vectors S_i and C_i in (A1) as follows,

$$z(t) = [a_{12}S_1 + a_{11}C_1, -a_{11}S_1 + a_{12}C_1, \dots, a_{m2}S_m + a_{m1}C_m, -a_{m1}S_m + a_{m2}C_m] c(t) \quad (\text{A.58})$$

Equivalently, (in matrix notation),

$$x(t) = QAc(t) \quad (\text{A.59})$$

where,

$$Q \triangleq [S_1, C_1, \dots, S_m, C_m] \in R^{N \times 2m} \quad (\text{A.60})$$

$$A \triangleq \begin{bmatrix} \underline{A_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \underline{A_m} \end{bmatrix} \in R^{2m \times 2m} \quad (\text{A.61})$$

and $A_i \in R^{2 \times 2}$ is defined as in (A.9). Hence, by inspection of (A.51) and (A.59), the matrix \mathcal{X} is given as,

$$\mathcal{X} = QA \quad (\text{A.62})$$

Squaring up \mathcal{X} and using (A.62) gives the confluence matrix,

$$\mathcal{X}^T \mathcal{X} = A^T Q^T Q A \triangleq M \quad (\text{A.63})$$

From the structure of (A.60)-(A.62), the components blocks of \mathcal{M} can be computed as,

$$\mathcal{M}_{ij} = A_i^T \begin{bmatrix} S_i^T S_j & S_i^T C_j \\ C_i^T S_j & C_i^T C_j \end{bmatrix} A_j \quad (\text{A.64})$$

Result (A.54) follows by substituting the identities (A.27)-(A.30) of Lemma A.3 into (A.64) and simplifying using expressions (A.1)-(A.10). Result (A.55) follows by setting $i = j$ in result (A.54) and simplifying by using the relation $\mathcal{B}_N(0) = N$. ■

LEMMA A.4 *Define,*

$$A \triangleq M - D^2 = \{A_{ij}\} \in R^{2m \times 2m} \quad (\text{A.65})$$

$$A_{ii} \triangleq M_{ii} - \frac{N}{2} \alpha_i^2 \quad 1_{2 \times 2} \in R^{2 \times 2} \quad (\text{A.66})$$

$$A_{ij} \triangleq \mathcal{M}_{ij} \in R^{2 \times 2} \quad (\text{A.67})$$

where D^2 is defined by in (S.19) of Theorem 3.1, and the matrix M and its submatrices \mathcal{M}_{ij} , M_{ii} are defined by (A.52)(A.55) of Theorem A1.

Let the quantities α_i , A_i , R_{ij} , R_{ij} be defined by (A.6)-(A.9), and assume that all frequencies $\{\omega_i\}_{i=1}^m$ are drawn from a bounded tone set i.e., $\Omega(m, T, \underline{\nu})$ defined in Definition 3.1. Then the following properties hold,

P1. $\bar{\sigma}(A_i) = \alpha_i$,

P2. $\bar{\sigma}(R_{ij}) = 1$

P3. $\bar{\sigma}(R_{ij}) = 1$

P4. $\bar{\sigma}(\Delta_{ii}) \leq \frac{\pi \alpha_i^2}{2\nu}$

P5. $\bar{\sigma}(\Delta_{ij}) \leq \frac{\pi \alpha_i \alpha_j}{2\nu}$

PROOF: It follows from the definition of ξ in (A.8), that the variables a_{i1} , a_{i2} , α_i are related as

$$\alpha_i^2 = a_{i1}^2 + a_{i2}^2 \quad (\text{A.68})$$

For the proof, extensive use will be made of the definitions (A.2)-(A.10). Continuing,

Proof of P1:

$$\bar{\sigma}(A_i) = \lambda_{\max}^{\frac{1}{2}}(A_i^T A_i) = \lambda_{\max}^{\frac{1}{2}}\left[(a_{i1}^2 + a_{i2}^2) \cdot I\right] = \alpha_i \quad (\text{A.69})$$

Proof of P2:

$$\bar{\sigma}(R_{ij}) = \lambda_{\max}^{\frac{1}{2}}(R_{ij}^T R_{ij}) = \lambda_{\max}^{\frac{1}{2}}\left[(c_{ij}^2 + s_{ij}^2) \cdot I\right] = 1 \quad (\text{A.70})$$

Proof of P3:

$$\bar{\sigma}(R_{ij}) = \frac{1}{\lambda_{\max}^{\frac{1}{2}}(R_{ij}^T R_{ij})} \frac{1}{\lambda_{\max}^{\frac{1}{2}}(c_{ij}^2 + s_{ij}^2)} \left\| \begin{bmatrix} c_{ij} \\ s_{ij} \end{bmatrix} \right\|_2 = 1 \quad (\text{A.71})$$

Proof of P4:

$$\bar{\sigma}(\Delta_{ii}) = \bar{\sigma}(\mathcal{M}_{ii} - \frac{N}{2} \alpha_i \cdot I_{2 \times 2}) = \bar{\sigma} \left(\frac{1}{2} \mathcal{B}_N(\omega_i T) A_i^T R_{ii} A_i \right) \quad (\text{A.72})$$

$$\leq \frac{1}{2} |\mathcal{B}_N(\omega_i T)| \cdot \bar{\sigma}(A_i)^2 \leq \frac{\pi \alpha_i^2}{4\nu} \quad (\text{A.73})$$

where (A. 72) follows by (A.55) of Theorem A. 1; equation (A. 73) follows by P3 for $i = j$; and the last inequality follows by Lemma A.2 and property P 1 proved above.

Proof of P5:

$$\bar{\sigma}(\Delta_{ij}) = \bar{\sigma} \left(\frac{1}{2} \mathcal{B}_N((\omega_i - \omega_j)T/2) A_i^T R_{ij} A_j + \mathcal{B}_N((\omega_i + \omega_j)T/2) A_i^T R_{ij} A_j \right) \quad (\text{A.74})$$

$$\leq \frac{1}{2} \bar{\sigma}(A_i) \cdot \bar{\sigma}(A_j) \left(|\mathcal{B}_N((\omega_i - \omega_j)T/2)| \cdot \bar{\sigma}(R_{ij}) + |\mathcal{B}_N((\omega_i + \omega_j)T/2)| \cdot \bar{\sigma}(R_{ij}) \right) \quad (\text{A.75})$$

$$= \frac{\pi \alpha_i \alpha_j}{2\nu} \quad (\text{A.76})$$

Here, (A.74) follows from (A.54) of Theorem A.1 and (A.67); and (A.76) follows by Lemma A.2 and properties P 1, P2, and P3 proved above. \blacksquare

LEMMA A.5 Let $X = XT \in \mathbb{R}^{2m \times 2m}$ be a symmetric *matrix* partitioned into 2×2 blocks, i.e.,

$$\mathbf{X} = \begin{bmatrix} X_{11} & \cdot & \cdot & \cdot & X_{1m} \\ & \ddots & & & \vdots \\ X_{m1} & \cdot & \cdot & \cdot & X_{mm} \end{bmatrix} \in \mathbb{R}^{2m \times 2m}; X_{ij} \in \mathbb{R}^{2 \times 2} \quad (\text{A.77})$$

Then,

$$\bar{\sigma}(X) \leq m \cdot \max_{ij} \bar{\sigma}(X_{ij}) \quad (\text{A.78})$$

PROOF: Let $\mathbf{x} \in \mathbb{R}^{2m}$ be a vector partitioned compatibly with X , i.e., $\mathbf{x} = [x_1^T, \dots, x_m^T]^T$, $x_i \in \mathbb{R}^2$. It will be useful to define the matrix $\Gamma_k \in \mathbb{R}^{2m \times 2m}$ which is the block diagonal matrix defined by,

$$\Gamma_k = \text{diag} \left[X_{k1}^T X_{k1}, \dots, X_{km}^T X_{km} \right] \quad (\text{A.79})$$

Consider the inequality,

$$(X_{ik}^T x_i - X_{kj} x_j)^T (X_{ik}^T x_i - X_{kj} x_j) \geq 0 \quad (\text{A.80})$$

Expanding (A.80) and using symmetry $X_{ij}^T = X_{ji}$ gives upon rearranging,

$$x_i^T X_{ki}^T X_{ki} x_i + x_j^T X_{kj}^T X_{kj} x_j \geq 2x_i^T X_{ik} X_{kj} x_j \quad (\text{A.81})$$

Continuing,

$$\bar{\sigma}^2(X) \triangleq \max_{\|x\|=1} x^T X^T X x = \max_{\|x\|=1} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m x_i^T X_{ik} X_{kj} x_j \quad (\text{A.82})$$

$$\leq \frac{1}{2} \max_{\|x\|=1} \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m x_i^T X_{ki} X_{ki} x_i + x_j^T X_{kj} X_{kj} x_j \quad (\text{A.83})$$

$$= m \max_{\|x\|=1} \sum_{j=1}^m \sum_{k=1}^m x_j^T X_{kj}^T X_{kj} x_j = m \max_{\|x\|=1} \sum_{k=1}^m x^T \Gamma_k x \quad (\text{A.84})$$

$$\leq m \sum_{k=1}^m \max_{\|x\|=1} x^T \Gamma_k x = m \sum_{k=1}^m \bar{\sigma}(\Gamma_k) \quad (\text{A.85})$$

$$= m \sum_{k=1}^m \max_i \bar{\sigma}^2(X_{ki}) \leq m^2 \max_{i,k} \bar{\sigma}^2(X_{ki}) \quad (\text{A.86})$$

Here, (A.82) follows by the partitioning in (A.77); equation (A.83) follows from inequality (A.81); equation (A.84) follows by the definition of Γ_k in (A.79); equation (A.85) follows by maximizing separately over each term in the summation; and (A.86) follows by the block diagonal structure of (A.79). Taking the square-root of (A.86) gives the desired relation (A.78). ■

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